COMUNICACION

ON THE HOMOGENEOUS FORM OF THE DIFFERENTIAL EQUATIONS OF RELATIVE MOTION OF THE THREE-BODY PROBLEM

Carlos A. ALTAVISTA

Observatorio Astronómico de La Plata

ABSTRACT

It has previously been shown (Celestial Mechanics, Vol. 6 N° 2, September 1972) that the system of differential equations of relative motion of the three - body problem can be brought into a homogeneous form. We give here an account of some additional proofs concerning the conditions of convergence of the geometric process used to obtain the new equations as well as a theorem due to Kolmogorov to consider analytical aspects of the problem.

1. It is well known that the differential equations of relative planetary motion, the Sun being the principal mass, can be written:

$$\text{(1)} \qquad \frac{d^2x}{dt^2} + k^2\ (m + m')\ \frac{x}{r^3} = k^2 \left(\frac{x' - x}{\Delta^3} - \frac{x'}{r'^3}\right) \qquad (x = x,y,z)$$

Similar corresponding differential equations can be written for the planet P'.

In a previous paper (Altavista 1972, hereafter mentioned as paper I), a geometric process has been devised to bring the system (1) into a homogeneous form. This transformation was achieved by treating in a convenient way the terms of the second members of equations (1) (Fig. 1).

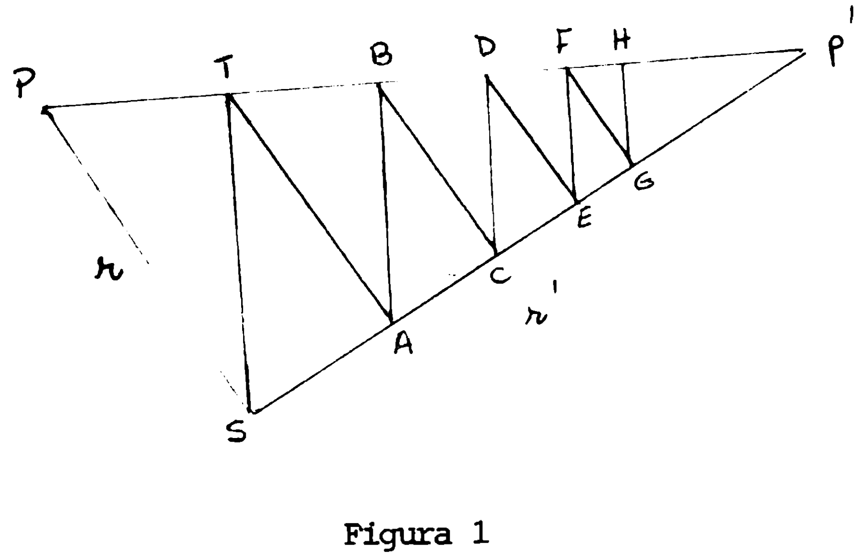

Figura 1

Here S is the Sun, P and P', the planets. We remember that we choose:

1) $ST \perp PP'$
2) $TA \parallel PS$
3) $AB \perp PP'$
4) $BC \parallel PS$, etc (A)

Next we shall consider the following questions:
i) The convergence of the infinite process suggested previously
ii) The justification that the reciprocal $1/\Delta$, where:

$$\Delta^2 = r^2 + r'^2 - 2\, r\, r' \cos H \,, \tag{2}$$

can be represented as a power series of $1/r$ (respectively $1/r'$). The respective indirect portions of the disturbing functions of P and P'

P : $x : r^3$; P' , $x' : r'^3$

can also be transformed within the framework of the proofs used to transform $1/\Delta$.
iii) The justification that the geometric process involved is equivalent to an integral over Δ.

i) The question of convergence of Process (A)

We remember first that equations (1) were finally transformed to the following form:

$$\frac{d^2x}{dt^2} + \Theta\,(1/r) \quad x = 0, \qquad (x = x,y,z) \tag{3}$$

where

$$\Theta\,(1/r^n)$$

is a power series in the reciprocals $1/r$ or $1/r'$, respectively. These new equations are valid together with the known formulas:

$$r = a\,(1 - e \cos E) \quad , \qquad r' = a'\,(1 - e' \cos E') \,, \tag{4}$$

where the angles E and E' (eccentric anomalies), give the orientation of the sides r and r' of the triangle PSP'. Let us now consider the chain of segments

(B) PS, TA, BC, DE, FG, ... WZ;

here WZ is the n parallel of the system. These segments will be indicated with (successively):

$$x_1 \,,\, x_2 \,,\, x_3 \,,\, \dots\, x_n \;.$$

Evidently:

$$x_2/x_1 = \theta_1 < 1$$
$$x_3/x_2 = \theta_2 < 1$$
$$x_4/x_3 = \theta_3 < 1$$
$$\dots\dots\dots$$
$$x_n/x_{n-1} = \theta_{n-1} < 1$$

By multiplying these inequalities we obviously obtain:

$$x_n/x_1 = \theta_1\ \theta_2\ \theta_3\ \dots\ \theta_{n-1} \quad << 1$$

A "fortiori" when $n \to \infty$, the above relationship, tends to zero. Let us now consider the segments: TS, BA, DC, FE, ...; let y_i be the i-th segment of this sequence. It is clear that:

$$\lim y_n/y_i \to 0.$$

It is then clear that the respective remainders of both sets of segments above considered are null sequences.

Next we consider the set of segments over the side Δ: PT, TB, BD, ... Calling z_i the nth segment of this series, we can write, by considering the set of rectangular triangles: PTS, TBA, etc.:

$$z_n^2 = x_n^2 - y_n^2 \quad , \tag{C}$$

from which follows that this last set of segments is also uniformly convergent. Then, process (A) tends to point P uniformly.

2. The second important question is connected to the transformation of the reciprocal $1/\Delta$ as a power series of $1/r$ (or $1/r'$ respectively). Since Δ^2 is given by:

$$\Delta^2 = r^2 + r'^2 - 2\, rr' \cos H \quad , \tag{5}$$

it is clear that we have to consider here a process of transforming a function of several variables in terms of a series of only one variable ($1/r$ or $1/r'$ respectively). In the first place, we must point out that the angle H between the distances r and r' is replaced through the definitions:

$$\begin{aligned} r &= a\ (1 - e \cos E) \\ r' &= a'\ (1 - e' \cos E') \end{aligned} \tag{6}$$

Then, the angles E and E' are used to substitute H, and allow us to orientate the triangle (Fig. 2).

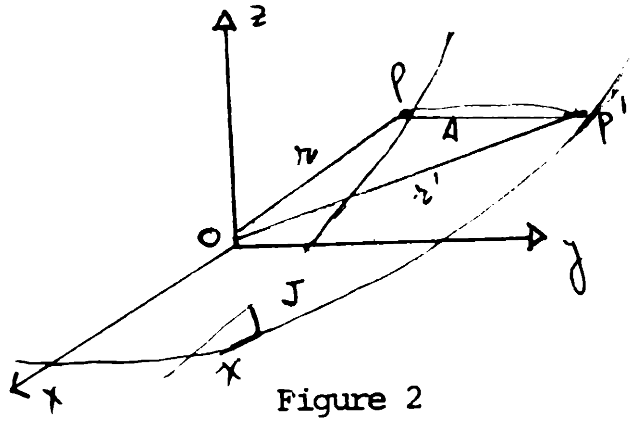

Figure 2

We then have to study the analytic conditions by which a function of two variables is transformed in a function of only one variable. In this sense, Kolmogorov and Sprecher have given theorems which can be applied in our particular case. Kolmogorov's theorem shows that a function of several variables can be represented by a superposition (a sum) of functions of only one variable. It is clear that, in general, such a transformation is not always possible, unless that the function to be transformed fullfils some special strong conditions.

Let $f(x, y)$ be a function of two variables x, y. Kolmogorov has succeeded to show that a representation of the form:

$$f(x, y) = \sum g\ (\phi_q\ (x) + \lambda\ \phi\ (y))\ , \tag{7}$$

is possible. The function g depends on f, but $\phi\ (x)$ and $\lambda\ \phi\ (y)$ do not depend on $f(x, y)$.

Formula (7) reduces the function $f(x, y)$ to sums and superpositions (that is to say functions of functions) of functions of one variable g, $\lambda_p \phi_q$. Representation of $f(x, y)$ in formula (7) is finite. In our case we have just demonstrated that the geometric process described in paper I can be reduced to a finite number of steps. It then proceeds to give the conditions under which Kolmogorov has demonstrated the validity of formula (7).

Such conditions are:

1) Terms of the right-hand member in formula (7) must be continuous and derivable.
2) The set of functions of the second member in formula (7) must be defined algebraically over a closed set of disjoint intervals.
3) The set of functions of the second member of formula (7) must approach the given original function monotonically. Such set must be increasing in a strict sense.

Let us now see if these conditions are fulfilled by the process stated in paper I.

Condition 1) is obviously satisfied, since each segment over the finite part of Δ is represented by a polynomial.

Segments over Δ form a chain of disjoint segments. This set tends to the limit P' monotonically. Since condition 1) is valid, condition 2) is also satisfied in our case. The set increases monotonically: in fact a simple rearrangement of the geometric process shows that the third condition is also fulfilled by the statements given in paper I.

3. The last interesting question is related to the nature of the geometric process devised in paper I to represent the function $1/\Delta$. However, it must be pointed out that our geometric process is also variable with the time t. The interpretation is facilitated if we consider first some fix time t. In these circumstances, the geometric device stated in paper I consists in a exhausting process the principles of which are similar to the one used by Archimedes to find the area under the parabola. He used a similar method to obtain the sum of the series

$$1 + \frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{4}{3}$$

Then, for a fix time t, our method implies a true integration over the interval corresponding to the side Δ. But side Δ changes with time, and a second process of integration is involved indeed.

To see this, let us refer our triangle S P P' to a conveniently chosen rectangular system of reference (Fig. 3):

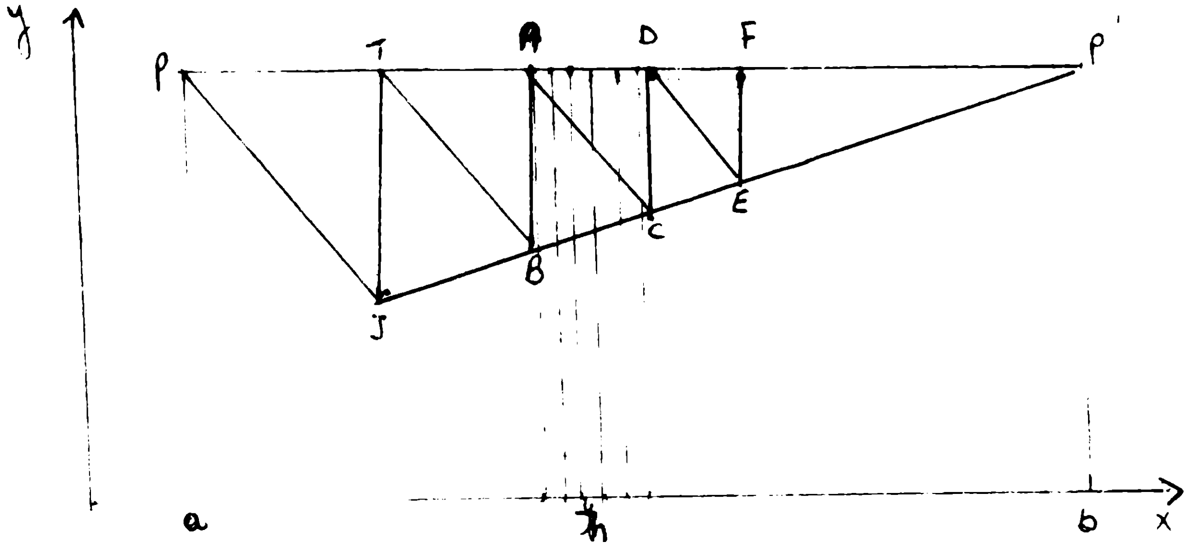

Figure 3

The x, y plane of reference moves in space together the S P P' triangle. In the figure, points P, A, C, E, ... over the side Δ have been projected on the x axis. In the next step we divide segments over Δ by a set of parallels to the y axis. Let us now define (Volterra) the following homogeneous algebraical form of j-th degree, corresponding to the j-th interval over (a, b). Write then:

$$(8) \qquad \Theta_j = \sum_m \sum_n \ldots \sum_t a_{m,n,\ldots t}\, y_m\, y_n \ldots y_t .$$

Let now h_i be any interval of the subdivision on (a, b). Take x_i the value of x within the interval h_i (i = 1, 2, 3 ...). Put

$$(9) \qquad a_{m,n,\ldots t} = \psi\,(x_m, x_n, \ldots x_t)\, h_m h_n \ldots h_t ; \; y_r = \phi(x_r) ;$$

we get:

$$(10) \qquad \Theta_j = \sum_m \sum_n \ldots \sum_t (x_m, x_n, \ldots x_t)\ \theta(x_m)\ \theta(x_n) \ldots \theta(x_t)\ h_m h_n \ldots h_t$$

In the next step, we decrease the lengths of the sets h_i over the j-th interval of (a, b). When this is done we obtain in the limit: (Volterra)

$$\lim \Theta_j = F \left| \left[\phi_a^b (x) \right] \right| , \tag{11}$$

where the right hand side is represented in the limit by:

$$F \quad \phi(x) = \int_a^b \int_a^b \dots \int_a^b \psi\,(x_m, x_n, \dots x_t) \prod_{i=1}^{t} \phi(x_i)\, dx_i \tag{12}$$

Obviously this process can be approached on every of the n chosen segments corresponding to the interval (a, b). These functional relationships appear because the plane of the triangle moves in space with the time t. This causes that points T, B, D, F ... to move continuosly over the side Δ. The exhausting process of integration, and the Volterra's process are clearly independent. In this sense, the representation of 1/Δ in terms of integrals shows to be very complicated. The fact that a synodical system of coordinates has been introduced leads to the interesting question about the nature of such systems. In this connection differential geometry provides an adequate mathematical tool to the study of moving axes of reference.

Let finally point out that G. Birkhoff has already described rotating surfaces (with uniform speed) round a fixed axis, in such a way that the potential field is carried by that surface.

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